The Role of Mathematical Modeling in Understanding the Groundwater Pollution

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Abstract

To understand the process of groundwater contamination diffusion and spreading, mathematical model are usually used. In the present study, a mathematical model based on the transport diffusion will be discussed. The equation is governed by several parameters including the water velocity, permeability and diffusion rate. Fourier transform is employed to obtain the exact solution of the problem. The mathematical model will be totally analyzed, programmed and tested using values for the parameters that simulate the actual data. Results suggests the model can bring some insights about the diffusion of the water contaminants as time progress from few days to hundreds of days.

Keywords: Mathematical modeling, underground water pollution, transport equation

1. Introduction

Groundwater is the supply of freshwater found beneath the earth's surface; It is one of our most valuable natural resources. Groundwater provides about 97 percent of the world's total supply of drinkable water. Residents rely on groundwater as their source of drinking water either through public utilities or individual residential wells. Clean, plentiful groundwater is a key to our health and way of life. However, the dangers to groundwater are countless since many of our activities on land affect the groundwater below. Pollution from the land's surface puts some groundwater at considerable risk of contamination. Contaminants are generally dissolved and carried by infiltrating rain water into unsaturated soil above the water table. From there, contaminants can enter into the saturated zone and begin to migrate in the direction of groundwater flow. Since groundwater advances so slowly, pollutants that get into it are not quickly diluted or flushed out. Also, it is difficult to detect groundwater pollution until it reaches a well, basement, underground utility or surface water area. By this time, pollution can be widespread. In addition, once groundwater becomes polluted, it is extremely difficult and expensive to clean up, even partially.

The contamination sources may be natural impurities such as magnesium, calcium, and chlorides, which may be picked up while the groundwater is moving through sedimentary rocks and soils. Or it may be agricultural sources such as pesticides, fertilizers, herbicides and animal waste. It might be due to the storage of agricultural chemicals near conduits to groundwater, or in uncovered areas or unprotected from wind and rain.

Other contamination sources may be industrial sources; it occurs when used water is returned to the hydrological cycle. Or when disposal of wastes associated with industrial activities are stored in an area close to groundwater bed. The other possible source of underground water contamination is the residential source; such as wastewater systems which can be a source of many categories of contaminants, including bacteria, viruses, nitrates from human waste, and organic compounds.

Despite of the extremely safe proofing technologies used in protecting waste depository plants, and all safety precautions for pipe lines carrying pollutant (chemicals and combusting gases) accidents and malfunctions in the systems can occurs which intern will causes a major threat for the environment and the human health. A quit large amount of experimental and theoretical studies addressing such a critical and important topic are available in the literature.

Ewing et al 1999; studied the groundwater flow driven by different pressure under saturated and unsaturated conditions. They presented numerical techniques for molding multi component gas flow in porous media. They utilized the mixed finite element method over quadrilaterals as a solver to the non-Darcy flow equation and a conservative Godunov-type scheme for the mass balance equations.

Schwarz et al (2009), in their work they assets the risk on environment and heath resulted from the leakage of CO2 in air near the ground, by having an exact solution of the advection–

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diffusion equation. Their results show a good agreements with numerical simulation performed for similar case with different boundary conditions.

Katalin et al (2002), utilized a relatively simple transport equation modeling to give a reliable forecast of the pollutants spreading in ground water after a mismatch, and to determine the level of possible damage to the environment. The solution of the approximated transport equations was used to describe the concentration of pollutants. Knowledge of the concentration level is a prerequisite for finding efficient ways to prevent the further spreading and the removal of pollutants from the groundwater.

Nagai et al 2004, Studied the formation and circulation of gases such as CO_2 and N_2 in the atmosphere-soil-vegetation system in nature. The subject of these studies is the behavior of one dimensional vertical formation and circulation of gas on the terrestrial surface and not the three-dimensional diffusion behavior from the point of leakage within a range as wide as a road.

Pommer et al 1999, a numerical using an operator-splitting method to couple advection-dispersive transport of organic and inorganic solute with geochemical equilibrium package. one dimensional multi-component model accounting for transport inorganic equilibrium chemistry. The work focus on validation of the experimental data to presented model. The model was modified to address the three dimensional case in order to compare to realistic field scale.

The convection –dispersion transport model is used in the present study to model the concentration distribution of pollutant in ground water due to malfunction of the water proofing system in waste material sewage compositing plant. The model solves the transport equation for a simplified case where all covering parameters were treated as constants. The source of pollutant is considered as a constant mass flow flux supplied along the span of the waste well. The model predicted the development of the concentration distribution for different position and times. The reverse seepage from air to ground is shown in the simulation to be very small, and the large difference between time scales suggests that air and ground can be modeled separately, with gas emissions from the ground model used as inputs to the air model.

2. The mathematical modeling

Assume that the waste material is stored in a cylindrical depository waste which is located along the z-axis. The depository is assumed to have a negligibly small cross section when compared to the size of the domain of investigation. The depository occupies the region 0<z<m.

The region of investigation of this study is given by: D={(x,y,z)| −∞<x<∞, , −∞<y<∞, 0<z<m}.

The source of contamination is a line of constant mass flow rate of a pollutant a long the z span from 0<z<m.

The concentration of the contaminant c(x,y,z,t) at any time is governed by the following transport equation:

$$Rn_0 C_t + v_x C_x = a_L v_x C_{xx} + a_T v_x C_{yy} + a_T v_x C_{zz} \qquad (1)$$
$$- \lambda Rn_0 C + q$$

Where v_x is the speed, R is the delay factor, a_L and a_T are the longitudinal and transversal disparity, respectively; λ is the Darcy rate and n_0 is the porosity. One of the major simplifications of the model is that all of these parameters are assumed constant.

The depository begins leaking at t = 0 with a constant rate q along the depth z. The initial condition $C(x, y, z, 0) = 0$ is assumed.

For the boundary conditions, we assumed that at the top boundary $z = m$, the waste material is discharged into the depository at a constant rate k where $k < q$, while the bottom is totally non-preamble. These two boundary conditions are written as:

$$C_z(x, y, m, t) = k \text{ , and } C_z(x, y, 0, t) = 0 \qquad (2)$$

The concentration satisfying the above conditions is assumed similar along the z-axis. Accordingly, we can assume that the concentration function C is a function of the three variables x, y and t only. An exact solution of the above equation can be obtained using the standard Fourier transform.

Now to solve Eq. (2) for the concentration function $C(x, y, t)$, we use the 2D-Fourier transform:

$$F\{f(x,y)\} = \hat{f}(\xi_1, \xi_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{i(x\xi_1 + y\xi_2)} dxdy$$

with inverse

$$F^{-1}\{\hat{f}(\xi_1, \xi_2)\} = f(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi_1, \xi_2) e^{-i(x\xi_1 + y\xi_2)} d\xi_1 d\xi_2.$$

Taking the Fourier transform of (2), $\tilde{C}(\xi_1, \xi_1, t) = \cdot (C(x,y,t))$, one obtains:

$$Rn_0 \tilde{C}_t - i\zeta_1 v_x \tilde{C} = -\zeta_1{}^2 a_L v_x \tilde{C} - \zeta_2{}^2 a_L v_x \tilde{C} - \lambda Rn_0 \tilde{C}$$
$$+ \frac{q}{2\pi} \qquad (3)$$

which can be written as a first-order ODE in the time t:

$$\tilde{C}_t + B\tilde{C}$$
$$= \frac{q}{2\pi Rn_0} \qquad (4)$$

where

$$B = \frac{-i\xi_1 v_x + \xi_1^2 \alpha_L v_x + \xi_2^2 \alpha_T v_x + \lambda Rn_0}{Rn_0}$$

Using the standard techniques for the solution of the first order ODE's, the solution of (4) is given as:

$$\tilde{C}(\xi_1, \xi_2, t) = \frac{q}{Rn_0} \int_0^t e^{-B(t-\tau)} d\tau$$

Then taking the inverse Fourier transform leads to:

$$C(x,y,t) = F^{-1}\{\tilde{C}(\xi_1,\xi_2,t)\} = \frac{q}{2\pi R n_0}\int_0^t F^{-1}\{e^{-B(t-\tau)}\}d\tau.$$

which leads to:

$$C(x,y,t) = \left(\frac{1}{4\pi v_x\sqrt{\alpha_L\alpha_T}}\right)q\int_0^t \frac{e^{-\lambda(t-\tau)}}{t-\tau}exp\left\{-\frac{Rn_0}{4v_x(t-\tau)}\left[\frac{\left(x-\frac{v_x(t-\tau)}{Rn_0}\right)^2}{\alpha_L}+\frac{y^2}{\alpha_T}\right]\right\} \tag{5}$$

which is the exact solution of the problem. The method was employed in the work of Allan *et al* [7]. It was used to analyze the fate and transport of underground water pollution in the case of short time depository of the contaminated material into the ground. The study was conducted using artificial values of the parameters which simulate the real values. The same values will be used here for the sake of testing.

3. Results and discussion

The solution of Eq. (1) which is given by Eq. (5) is tested for the standard values of the parameters given in [7].

Figure 1 represents the contamination on the region D= {(x, y)| 0≤x≤1200, -100≤y≤100} at different time levels. It is clear that the contamination is spreading on a wide area as time goes on. Since the leaking is continuous, the contamination is carried out by the underground water to other uncontaminated areas.

Figure 2 represents snapshots of the concentration contours at different time intervals along the x-y plane z=0 . The maximum concentration is the same while the contaminated area is spreading. More clear picture of the spreading of the contamination is show in Figure 3 which is a snap shot of the concentration C(x, 0, t) when z=0 at different time levels. The results indicate that the highest level of concentration is in the premises of the source line and it is dying out as it spread in the x-y plane with time. The decay in concentration is dropped as $x^{-\beta}y^{-\alpha}$.

The results presented in this study suggest that the model is adequate to study the fate and transport of underground water contamination and can be applied at different regions of the UAE.

4. Conclusions

A mathematical model based on the convection–dispersion transport equation with constant properties assuming constant source of waste flux, is used to model the contamination distribution of the pollutant into the soil. The computationally inexpensive dynamical model is meant to be used as a tool to predict the contamination spreading in the soil in case of unfortunate incidence of leakage. The results thus obtained, indicates that it will take a very long time before the contaminated material to disappear. It also indicates that as time goes on, the contaminated material will continue to spread over a larger area but with less concentration. It is possible to reformulate the model to include other cases such as variable coefficients.
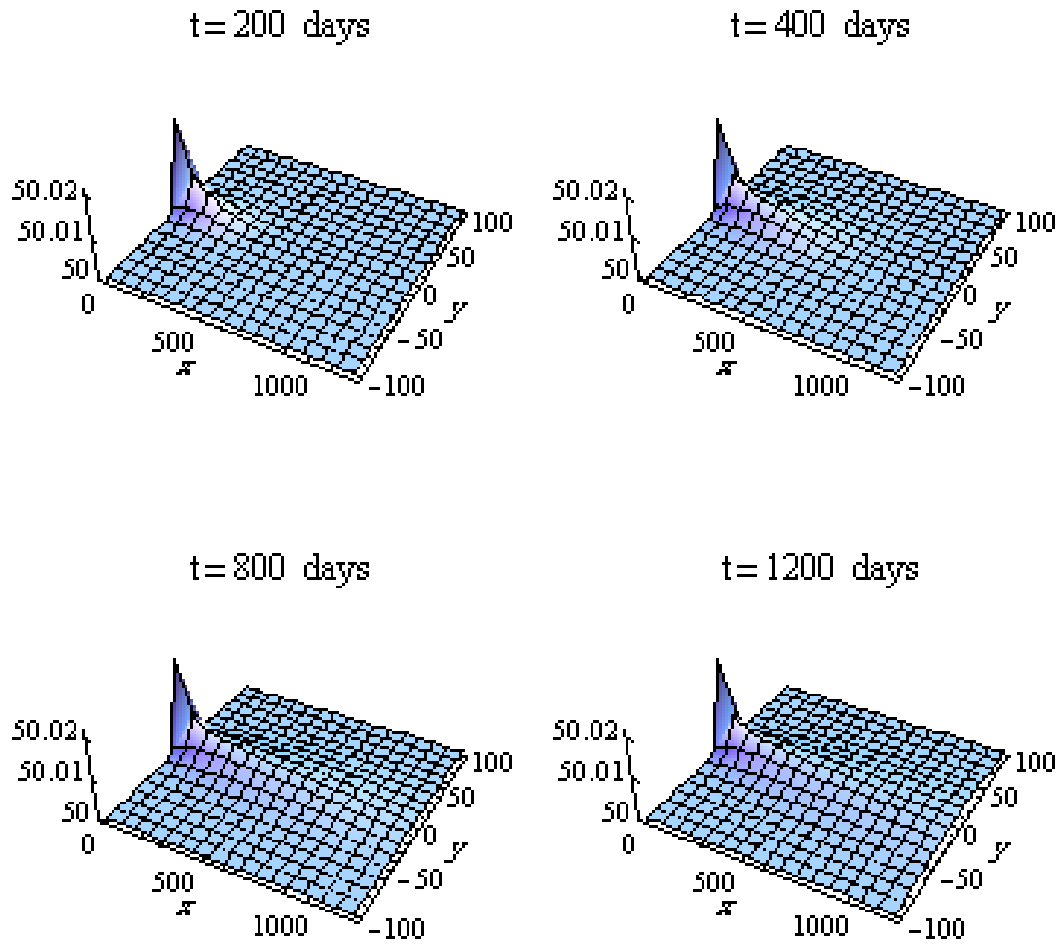
Fig. 1.   **Surface plots of the concentration function C(x,y,t) on the region D={(x,y)| 0<x<1200, -100<y<100}, z=0 at different time levels**
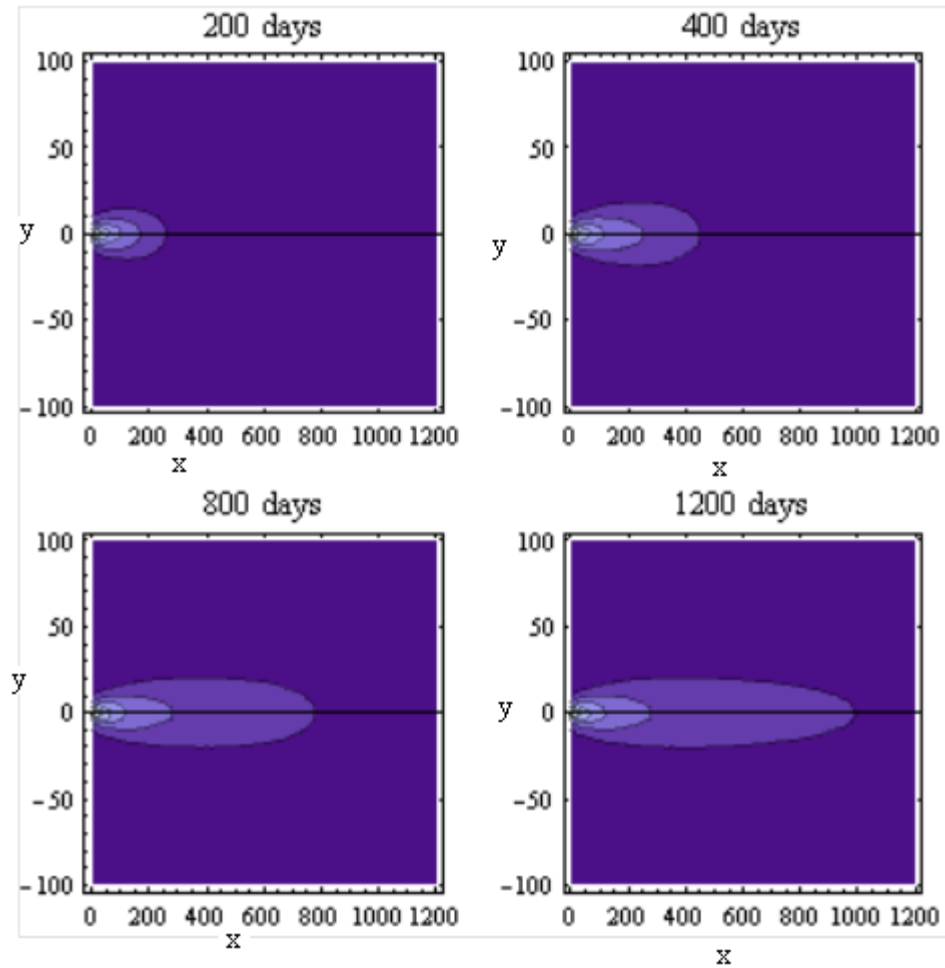
Fig. 2. Contour plots of the surfaces of Figure 1 showing the spreading of the waste material as time goes on
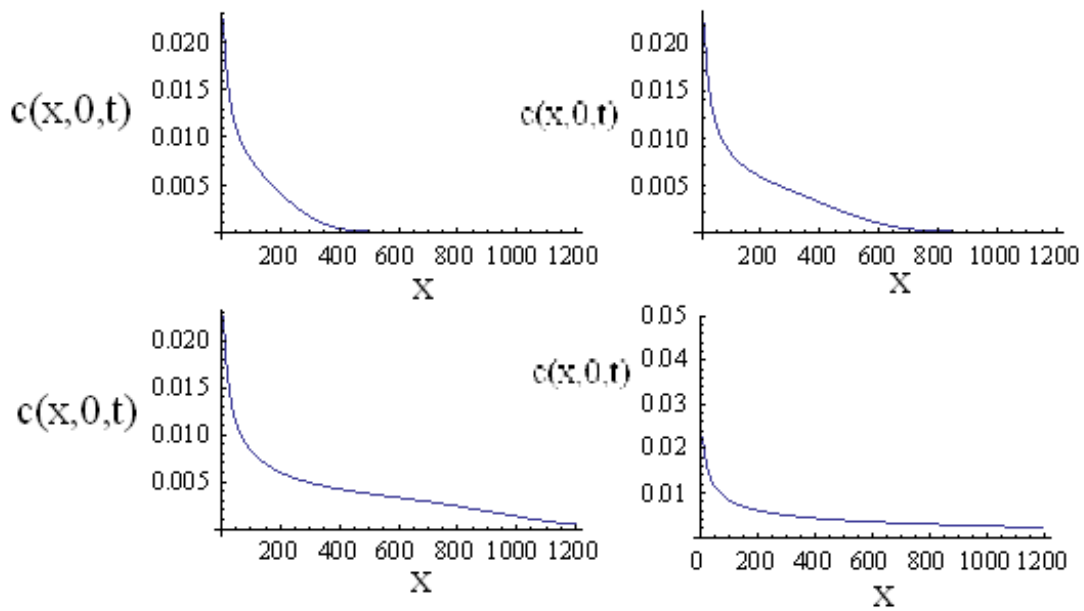

Fig. 3. Curves of the concentration of the waste at y =0 along the x direction at the time levels of Figure 1.

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